Local existence of solutions for the modified Korteweg-de Vries (mKdV) equation in weighted Sobolev spaces

DOI: 10.46932/sfjdv2n1-034

Received in: November 1st, 2020
Accepted in: December 30th, 2020

Milton M. Cortez Gutiérrez
Department of Mathematics
National University of Trujillo-(UNT) , Trujillo, Peru
E-mail: mcortezgutierrez@yahoo.es

Hernan O. Cortez Gutiérrez
Universidad Nacional del Callao (UNAC), Callao, Perú
E-mail: hernanoscarcortezgutierrez@gmail.com

Girady I. Cortez Fuentes Rivera
Universidad Nacional del Callao (UNAC), Callao, Perú
E-mail: giacortezf@gmail.com

Liv J. Cortez Fuentes Rivera
Universidad Privada San Juan Bautista (UPSJB), Lima, Perú
E-mail: livjoisc@gmail.com

Deolinda E. Fuentes Rivera Vallejo
Universidad Inca Garcilaso de la Vega, Lima, Perú
E-mail: deitofuentesriva@outlook.es

ABSTRACT
Our goal in this paper is to give a proof of local existence of solutions for the modified Korteweg-de-Vries (mKdV) in a weighted Sobolev space with initial data. A parabolic regularization is suggested, discussed and implemented for the approximation of solutions for the mKdV equation with an appropriate class of nonlineariy. The study for the mKdV equation and computations of blow-up without weighted was treated by Bona,J.L and Dougalis,V.A.. [4] Further results can be found in Bona,J.L.-Souganidis,P.E. and Strauss,W.A.[6] .
Connections with biophysics are illustrated in Cortez, H et al [27]-[29].

Keywords: Korteweg-de Vries, weighted Sobolev, deformation of the eyeball, dna breathing, helical spaces curves of Lamb.

1 INTRODUCTION

The purpose of this paper is to give a proof to the local existence of solutions for the modified Korteweg-de Vries (mKdV) equation in a weighted Sobolev space. It is worthwile to notice that the mKdV equation describes a time evolution for the curvature of certain types of helical space curves (see Lamb,G. [18]).

Foremost, we introduce the initial value problem for the modified Korteweg-de Vries (mKdV) equation as being

$$\partial_t u + 3u^m \partial_x u + h\partial_x^3 u = 0 \ , \ t > 0 \ , \ x \in \mathbb{R} \quad , \text{m=2, h=1} \qquad (1.1)$$
$$u(x,0) = u_0(x) \quad , \quad x \in \mathbb{R} \qquad (1.2)$$

Where $u: [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $u_0: \mathbb{R} \to \mathbb{R}$ is a given function

The initial-value problem for (1.1) and (1.2) is known to be locally well-posed in reasonable function classes (cf. Iorio,R.[14],Kapitula,T[15] or Weinstein,M. [25]).

The linear form:

$$\partial_t u + h\partial_x^3 u = 0 \ , \ t > 0 \ , \ x \in \mathbb{R} \qquad \text{(linear form)}$$

presents a solution with Fourier methods.

The plan of the paper is as follows. In the next section, one introduces weighted Sobolev space and an m-accretive operator is presented and theory relating to the underlying partial differential equations is outlined. A method according to Hille-Yosida is applied as well. Section 3 we give the main result about the existence of solution for the modified Korteweg-de Vries (mKdV) equation in weighted sobolev space.

This work illustrates applications of Lax-Milgram lemma that also have applications in biomechanics of the human body [26]. Keeling analyzes the deformation of the eyeball.

About the solutions of non-linear partial differential equations analyzed are similar to the solutions found for DNA vibrations (DNA Breathing) [27]-[30].

We conjecture that Lamb's theoretical basis [18] helps to understand the dynamics of DNA breathing

2 M-ACCRETIVE OPERATORS IN HILBERT SPACES

In this section we establish an useful definition of m-accretive operators in Hilbert spaces. Let A be an operator defined in a Hilbert space H for the scalar product ((,)) and equipped with the norm $\| \quad \|$, with domain $D(A)$. We say that the operator A is accretive in H if

$$\|u + \lambda A u\| \geq \|u\|,$$

for all $u \in D(A)$ and all $\lambda > 0$.

Hence, we say that an operator A in a Hilbert space H is m-accretive if the following holds

i) A is accretive

ii) For all $\lambda > 0$and all $f \in H$, there exists $u \in D(A)$ such that

iii)

$$u + \lambda A u = f$$

Which is an underlying partial differential equation. It follows easily from the definition that if A is an m-accretive operator in H, the mapping $f \mapsto u$ is a contraction $H \to H$, and is one to one $H \to D(A)$ more precisely the above mapping is denoted by $J_\lambda(A)$, or $(I + \lambda A)^{-1}$. We have $J_\lambda \in \mathcal{L}(H)$, $\|J_\lambda\|_{J_\lambda} \leq 1$, and $R(J_\lambda) = D(A)$. J_λ is called the resolvent of A and A_λ is the Yosida approximation of A, defined by $A_\lambda = \lambda^{-1}(I - J_\lambda)$. It is clear that the graph $G(A)$ is closed in $H \times H$, $D(A) \hookrightarrow H$.

The notation to be used is mostly standard. We first introduce the space $\mathcal{S}$ of rapidly decreasing functions on $\mathbb{R}^+$

$$\mathcal{S}(\mathbb{R}^+) = \left\{\varphi \in C^\infty(\mathbb{R}^+);\ \sup_{x\in\mathbb{R}}\left|x^\beta D^\alpha \varphi(x)\right| < \infty\,,\ \ \forall\, \alpha, \beta\ \in \mathbb{N}\right\}$$

The functions $e^{-x^2}p(x)$ where $p(x)$ is a polynomial and $\varphi \in C_0^\infty(\mathbb{R}^+)$, are standard examples of rapidly decreasing functions.

It is readily seen that $\mathcal{S}$ is a locally convex, linear topological space with the topology defined by the family of seminorms

$$|\varphi|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^+}\left|x^\beta D^\alpha \varphi(x)\right|$$

Furthermore, $C_0^\infty(\mathbb{R}^+)$ is a dense linear subspace of $\mathcal{S}$.

In order to study the initial value problem (1.1) and (1.2), we wish now to introduce the weighted Sobolev space.

We put $H_r^s(\mathbb{R}^+)$ as being a subspace of $\mathcal{S}(\mathbb{R}^+)$ which is a Hilbert space with the inner product $(u, v)_{r,s} = (M_r\Lambda^s u, M_r\Lambda^s v)_2$. Here $(.,.)_2$ is the inner product in $L^2(\mathbb{R}^+)$. For $r \in \mathbb{R}^+$, let Λ^s be the Pseudo-differential operator defined by $\Lambda^s = (I - \partial_x^2)^{s/2}$. Furthermore, we set $M_r u(x) = (1 + |x|^2)^{r/2} u(x)$.

If $r_1 \geq r_2$, $s_1 \geq s_2$ one can obtain $H_{r_1}^{s_1}(\mathbb{R}) \subset H_{r_2}^{s_2}(\mathbb{R})$, with continuous injection.

In addition, if $u \in H^{s_1}_{r_1}(\mathbb{R})$ and $v \in H^{s_2}_{r_2}(\mathbb{R})$ then $uv \in H^{s}_{r}(\mathbb{R})$ whenever $s_1, s_2 \geq s, s_1 + s_2 - s \geq 1/2$ and so there exists a constant $c > 0$ such that

$$|uv|_{r,s} \leq c\,|u|_{r_1,s_1}\;|v|_{r_2,s_2}$$

In order to apply semigroup theory, one can apply the parabolic regularization to the initial value problem (1.1) and (1.2) in such a way that it can be put in the form

$$\partial_t u + Au = F(u)\;,\; t > 0\,, \qquad (1.3)$$

$$u(0) = u_0\;, \qquad (1.4)$$

Where $F(u) = -3u^2\partial_x u$, without loss of generality we may consider the same notation $u = u_\mu$, $A = A_\mu = \partial_x^3 u - \mu\partial_x^2 u$, and $\mu > 0$. On the other hand, the functional $a: H^4_r(\mathbb{R}^+) \times H^4_r(\mathbb{R}^+) \to \mathbb{R}$ defined by $a(u,v) = \big((I + \lambda A)u, v\big)_{r,4}$ is bilinear on $H^4_r(\mathbb{R}^+) \times H^4_r(\mathbb{R}^+)$ and is said to be coercive if $a(u,u) \geq c|u|^2_{r,4}$,

$\forall u \in H^4_r(\mathbb{R}^+)$ for some constant $c > 0$. In addition, the functional a is said to be continuous if there exists $M > 0$ such that

$$|a(u,v)| \leq M|u|_{r,4}|v|_{r,4} \qquad (1.5)$$

Before proceeding to the proof of the main result, we establish some preliminary lemmas.

Lemma 2.1 The functional $a: H^4_r(\mathbb{R}^+) \times H^4_r(\mathbb{R}^+) \to \mathbb{R}$ is coercive on $H^4_r(\mathbb{R}^+)$.

Proof.

In fact, $a(u,u) = \big((I + \lambda A)u, u\big)_{r,4} = (u,u)_{r,4} + \lambda(Au,u)_{r,4}$

Letting $\varphi = \Lambda^4 u$, from the above equality we have $a(u,u) = (u,u)_{r,4} + \lambda(A\varphi,\varphi)_{r,0}$

By integration by part, it yields

$$a(u,u) = (u,u)_{r,4} - 2\lambda r\int_{\mathbb{R}^+}\frac{x}{(1+x^2)}M_r^2\partial_x\varphi\partial_x^2\varphi dx - \lambda\int_{\mathbb{R}^+}M_r^2\partial_x\varphi\partial_x^2\varphi dx -$$

$$-\lambda\mu\int_{\mathbb{R}^+}M_r^2\varphi\partial_x^2\varphi dx \geq -(2\lambda r + \lambda\mu)\int_{\mathbb{R}^+}M_r^2\varphi\partial_x^2\varphi dx + \frac{\lambda}{2}\int_{\mathbb{R}^+}(M_r^2)_x(\partial_x\varphi)^2 dx \geq$$

$$\geq |u|^2_{r,4} - (2r+\mu)\lambda r^2\int_{\mathbb{R}^+}M_r^2\varphi^2 dx = |u|^2_{r,4} - (2r+\mu)\lambda r^2|u|^2_{r,4}$$

Hence, $a(u,u) \geq (1-(2r+\mu)\lambda r^2)|u|_{r,4}^2$ we choose $1-(2r+\mu)\lambda r^2) > 0$.

We conclude this section arguing as the proof of **Lemma 2.1** the following lemma.

Lemma 2.2 The functional $a: H_r^4(\mathbb{R}^+) \times H_r^4(\mathbb{R}^+) \to \mathbb{R}$ is continuous. More precisely $a(u,v)$ verifies (1.5).

3 MAIN RESULT.

It follows from Lemma 2.1 and Lemma 2.2 that one can argue as above that the operator A is m-accretive in $H_r^4(\mathbb{R}^+)$. We can now apply from the theory of semigroups, Pazy,A.[20] that there exists $(T(t))_{t\geq 0} \subset \mathcal{L}(H_r^4(\mathbb{R}^+))$ the semigroup of contractions generated by A, i.e.

$$\|T(t)\|_{\mathcal{L}(H_r^4(\mathbb{R}^+))} \leq 1 \ , \ \forall\, t \geq 0 \qquad (1.6)$$

On the other hand, the operator $F: H_r^4(\mathbb{R}^+) \to H_r^4(\mathbb{R}^+)$ is globally Lipschitz continuous on $H_r^4(\mathbb{R}^+)$, in the sense that there exists a constant $L > 0$ such that

$$|F(v)-F(u)|_{r,4} \leq L|v-u|_{r,4} \quad , \text{ for all } \ u, v \in H_r^4(\mathbb{R}^+) . \qquad (1.7)$$

Therefore, given any $u_0 \in H_r^4(\mathbb{R}^+)$, there exists a unique global weak solution u of (1.3) and (1.4) in the sense that $u \in C([0,\infty), H_r^4(\mathbb{R}^+))$ and

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\,F\big(u(s)\big)ds \ , \quad \text{for all } t \geq 0 \qquad (1.8)$$

<u>**Theorem 3.1**</u>. Let u_μ be the unique global weak solution of (1.3) and (1.4), then there exists a function $u \in H_r^4(\mathbb{R}^+)$ such that $\lim_{\mu\to 0} u_\mu = u$, in addition the function u is the unique solution of (1.1) and (1.2).

<u>**Proof.**</u>

We set $V(t) = u_\mu(t) - u_\nu(t)$, $W(t) = u_\mu^2(t) + u_\mu(t).u_\nu(t) + u_\nu^2(t)$ and $U = u_\nu(t)$. It follows from the relation (1.3) that U, V and W hold

$$\partial_t V + \partial_x^3 V + \partial_x(VW) = (\mu-\nu)\partial_x^2 U + \mu\partial_x^2 V \qquad (1.9)$$

By multiplying of V the equation (1.9) in $H_r^4(\mathbb{R}^+)$, it yields:

$$\frac{1}{2}\frac{d}{dt}|\mathrm{V}(t)|_{r,4}^{2} = (V,(\mu-\nu)\partial_x^2 U) + \mu(V,\partial_x^2 V) - (V,\partial_x^3 V) - \big(V,\partial_x(VW)\big)$$

Hence, one obtains the following estimate for each inner product in $H_r^4(\mathbb{R})$, so that

$$\frac{1}{2}\frac{d}{dt}|\mathrm{V}(t)|_{r,4}^{2} \leq C_1|\mu-\nu| + C_2|\mathrm{V}(t)|_{r,4}^{2} \qquad (1.10)$$

Where C_1 and C_2 are positive constants.

By integration of the above inequality (1.10) from 0 to t, we get

$$|\mathrm{V}(t)|_{r,4}^{2} \leq C_1 t|\mu-\nu|e^{tC_2} \qquad (1.11)$$

So that the inequality (1.11) shows that the family $\big(u_\mu(t)\big)$ is a Cauchy sequence in $H_r^4(\mathbb{R}^+)$, this implies that $u_\mu(t) \to u(t)$, in $H_r^4(\mathbb{R}^+)$ as $\mu \to 0$.

Next, by making $Bu_\mu(t) = \mu\partial_x^2 u - \partial_x u_\mu^3 - \partial_x^3 u$, we can claim that

$$u_\mu(t) - u_\mu(\sigma) = \int_\sigma^t Bu_\mu(s)ds \qquad (1.12)$$

Furthermore, it is readily seen that the operator A is a linear continuous operator from $H_r^4(\mathbb{R}^+)$ to $H_r^0(\mathbb{R}^+)$, so that $|\mathrm{Au}|_{r,0} \leq C|\mathrm{u}|_{r,4}$

Evidently

$$\left|\langle Bu_\mu - Bu,\varphi\rangle\right| \leq C\left|\mathrm{A}u_\mu - \mathrm{Au}\right|_{r,0}|\varphi|_{r,0} + \left|\left(\partial_x u_\mu^3 - \partial_x^3 u,\varphi\right)_{r,0}\right|$$

Letting $\mu \to 0$, we deduce that $Bu_\mu \rightharpoonup Bu$, weak in $H_r^0(\mathbb{R}^+)$, it follows from Lebesgue's dominated convergence Theorem

$$\int_\sigma^t Bu_\mu(s)ds \rightharpoonup \int_\sigma^t Bu(s)ds \ , \quad \text{in} \quad H_r^0(\mathbb{R}^+) \quad \text{as } \mu \to 0. \qquad (1.13)$$

By using the relations (1.12) and (1.13), it yields

$$u(t) - u(\sigma) = \int_{\sigma}^{t} Bu(s)ds \quad , \quad \text{in} \quad H_r^0(\mathbb{R}^+)$$

Because $t \mapsto \quad |\mathrm{Bu(t)}|_{r,0}$ is measurable and belong to the space of $H_r^0(\mathbb{R}^+)$-valued Bochner integrable functions, we have that the function u is continuous absolutely from $[0, T]$ to $H_r^0(\mathbb{R}^+)$.Therefore

$$\partial_t u = -Bu(t) = -\partial_x u^3 - \partial_x^3 u \ , \quad \text{a.e.} \quad \text{in } [0, \mathrm{T}]$$

On the other hand, whereas the unique solution one verifies easily from Gronwall's Lemma.

4 CONCLUSION

It is worthwhile to be noted that the results achieved were strongly used the semigroup theory and some theorems relating the evolution equation so as well the m-accretive operator. On the other hand, beyond some inequalities like for instance Young's inequality, Gronwall's lemma and embedding's theorem in weighted Sobolev space were used for obtained some estimates to the solution of an approximated problem. Finally from Gronwall's lemma we readily claim that the uniqueness of solution to the modified Korteweg-de Vrie equation in weighted Sobolev space holds. Certainly we point out that the procedure through the multiplicative technics was good for applying the notion of weak convergences.

REFERENCES

[1] Adams, R. Sobolev space, Academic press N.Y., 1972.

[2] Alber,J. and Bona,J., Comparisons between model equations for long waves Journal nonlinear Sci 1(1991) pp. 345-374.

[3] Bisognin,V.,On the asymptotic behavior of the solutions of a nonlinear dispersive system of Benjamin-Bona –Mahonytype, Bolletino UMI 7 10-B(1996), pp. 99-128.

[4] Bona,J.L.-Dougalis,V.A.,-Karakashian,O.A and McKinney, Conservative,high- order schemes for the generalized Korteweg-de Vries equation, in preparation

[5] Bona,J.-Ponce,G.-Saut,J.C., A model system for strong interaction between internal solitary waves, Comm. Math. Phys. Appl.Math.,143(1992) pp. 287-313

[6] Bona,J.L. –Souganidis,P.E. and Strauss, W.A. ,Stability and instability of solitary waves of KdV type, Proc.Roy.Soc.London ser. A 411(1987) 395-412.

[7] Bona,J.L.and Ridgway Scott, Solutions of the Korteweg-de Vries equation in fractional order Sobolev spaces, Duke Math. J. **43** (1976), no. 1, 87–99

[8] Bona,J , Sun,S.M. and Bing Yu Zhan, A non-homogeneous boundary-value

[9] Bona,J and Ragnar Winther, problem for the Korteweg-de Vries equation in a quarter plane, Journal: Trans. Amer. Math. Soc. **354** (2002), 427-490

[10] Chen,G., A note on the boundary stabilization of the wave equation, Siam J. control and Optimization, 19(1)(1981),197-113

[11] Cortez G.M.M., Análisis Comparativo para la BBM en espacios de Sobolev con peso .Tesis Doctoral IMUFRJ, Brasil 1998.

[12] Ginibre,J., Y.Tsutsumi and G.Velo, Existence and uniqueness of solutions for the generalized Korteweg de Vries equation, Math. Z. **203** (1990), no. 1, 9–36

[13] Guidalia, J.M.- MarzocchiI, A. Longtime Behavior of Strongly Damped Wave Equations, Global Attactor and about the Dimension, SIMA J. Math. Anal. Vol, 22, N°4, (1991), 879-895.

[14] Iorio,R.,-Cardoso,-Djairo,G.,Lopes,O., Partial differential equation, Ed. Springer Verlag.1986

[15] Kapitula, Tood., on the stability of traveling wave in weighted L^∞ space, Journal of Differential equations, 1994.

[16] Kenig,C, G.Ponce and Vega L. ,well-posed ness of the initial value problem for the korteweg-de vries equation, journal of the american mathematical society volume 4, number 2, april 1991

[17] Kruzhcov,S.N. and Faminski, A.V. Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation, Mat. Sb. (N.S.) **120(162)** (1983), no. 3, 396–425 (Russian).

[18] Lamb,G.A., solitons on moving space curves, Journal Math.Phys,15 2157(1974).

[19] Medeiros,Luis A. Equacoes diferenciais parciais, IMURFJ, 1983

[20] Pazy,A., , Semigroup of linear operators and applications to partial differential equations, Springer Verlag N.Y. 1983

[21] Quinn,J. and RusselL,D.l., Asymptotic Stability and Energy Decay for solutions of hyperbolic equation with boundary damping, Proceedings of the royal society of Edinburg,77A(1987, 97-127

[22] Renardy,M. and Rogers,R.., An introduction to partial differential equations, Springer Verlag N.Y. 1993

[23] Ruan, W.H. One-parameter family of invariant sets for nonweakly coupled nonlinear parabolic systems, J. Math. Anal. Appl. **189** (1995), no. 3, 763–780

[24] Temam, R. Infinite. Dimensional Dynamical Systems in Mechanics and Physics, Springer Verlag (1988).

[25] Weinstein M., On the solitary wave of the generalized of KDV equation. Amer. Math, Soc. Lectures in applied Math. 23 (1986).

[26] Keeling S., Propst G, Stadler G, Wackernagel W . A mathematical model for the deformation of the eyeball by an elastic band. Medical University of Graz. 23 (2009).

[27] Cortez, H., Drigo Filho, E., Ruggiero, J. R., Breather stability in one dimensional Lattices with a symmetric Morse Potential, TEMA, Tend. Mat. Apl. Comput. 9 (2) (2008) 205-212. https://doi.org/10.5540/tema.2008.09.02.0205.

[28] Cortez, H., Drigo Filho, E., Ruggiero., J. R., Cortez, M., Mobile breathers in a nonlinear model for DNA breathing, Eclética Química Journal 42 (2017) 71-75. https://doi.org/10.26850/1678-4618eqj.42.1.2017.p71-75.

[29] Cortez, H., Tese de doutorado, Modelo Dinâmico e estatístico aplicado à transição de fase. UNESP.(2009)
https://repositorio.unesp.br/bitstream/handle/11449/100461/cortezgutierrez_ho_dr_sjrp.pdf?sequence=1&isAllowed=y.

[30] Cortez, H., Drigo Filho, E., Ruggiero, J. R., Gutierrez, M. C., Fuentes Rivera, L. C., Thermodynamics of DNA with "hump" Morse potential, Eclética Química Journal 41 (2016) 60-65. https://doi.org/10.26850/1678-4618eqj.v41.1.2016.p60-65.